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New Results in the Least p th Approach to Minimax Design

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Abstract—We present results of two general approaches for obtaining minimax designs through a sequence of least p th approximations demonstrating increased efficiency over previous least p th algorithms. Documented computer programs are available.

INTRODUCTION

This short paper demonstrates the acceleration of convergence to minimax solutions by extrapolation on a sequence of least p th solutions [1] with geometrically increasing values of p , and compares the results with an efficient extension of work by Charalambous

and Bandler [2], [3], in which a sequence of least p th solutions with finite values of p are obtained in an effort to reach a minimax solution. Documented computer programs are available [4], [5], as well as the theoretical background [6]–[8].

THEORY

We minimize, with respect to ϕ for given ξ and $p > 1$, the function

$$U(\phi, \xi, p) \triangleq \begin{cases} M(\phi, \xi) \left(\sum_{i \in K} \left(\frac{f_i(\phi) - \xi}{M(\phi, \xi)} \right)^q \right)^{1/q}, & M(\phi, \xi) \neq 0 \\ 0, & M(\phi, \xi) = 0 \end{cases} \quad (1)$$

where

$$M_f(\phi) \triangleq \max_{i \in I} f_i(\phi), \quad M(\phi, \xi) \triangleq M_f(\phi) - \xi, \\ q \triangleq p \operatorname{sgn} M(\phi, \xi)$$

and

$$K = \begin{cases} I \subset \{1, 2, \dots, m\}, & M(\phi, \xi) < 0 \\ J \triangleq \{i \mid f_i(\phi) - \xi \geq 0, i \in I\}, & M(\phi, \xi) > 0 \end{cases} \quad (2)$$

and where $\phi \triangleq [\phi_1 \phi_2 \dots \phi_k]^T$ is the design parameter vector, and $f_1(\phi), f_2(\phi), \dots, f_m(\phi)$ are m linear or nonlinear functions directly related to the response error functions such that if $M_f(\phi) > 0$ the specifications are violated and if $M_f(\phi) < 0$ the specifications are satisfied.

Charalambous has shown [6] that if we have u and ϕ such that

$$\sum_{i=1}^m u_i \nabla f_i(\phi) = 0 \\ \sum_{i=1}^m u_i = 1, \quad u_i \geq 0, \quad i = 1, 2, \dots, m \quad (3)$$

then, if $\sum_{i=1}^m u_i f_i(\phi)$ is convex with respect to ϕ ,

$$\sum_{i=1}^m u_i f_i(\phi) \leq M_f(\check{\phi}) \leq M_f(\phi) \quad (4)$$

where $\check{\phi}$ is the minimax optimum which is being sought,

$$u \triangleq [u_1 u_2 \dots u_m]^T$$

and

$$\nabla \triangleq [\partial/\partial\phi_1 \partial/\partial\phi_2 \dots \partial/\partial\phi_k]^T.$$

The conditions (3) are satisfied at each optimum point $\check{\phi}(p, \xi)$ for a least p th objective function, yielding

$$\sum_{i=1}^m u_i f_i(\check{\phi}(p, \xi)) \leq M_f(\check{\phi}) \leq M_f(\check{\phi}(p, \xi)) \quad (5)$$

where, assuming K contains all critical sample points,

$$u_i = \frac{v_i}{\sum_{i \in K} v_i} \quad (6)$$

$$v_i = \begin{cases} \left(\frac{f_i(\check{\phi}(p, \xi)) - \xi}{M(\check{\phi}, \xi)} \right)^{q-1}, & i \in K \\ 0, & i \notin K. \end{cases} \quad (7)$$

The first term of (5), under the stated conditions, is a lower bound on $M_f(\check{\phi})$. It is, at any least p th solution, an optimistic indication of the ultimate minimax error to be expected for a particular design.

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THE ξ ALGORITHM

An Implementation of Work by Charalambous [6]

ξ^{r+1} is set to the lower bound from the r th optimization. p is kept constant. We start at ϕ^0 by setting ξ^1 to an initial guess of a lower bound and setting $\xi^0 < \xi^1$ to a margin for discarding all functions f_i for which $f_i(\phi^0) < \xi^0$, which are considered unlikely to be active at ϕ . ξ^r is used as a level for discarding functions, so that

$$I^{r+1} = \{i \mid f_i(\phi^r) \geq \xi^r\}.$$

The functions specifically required for the least p th objective and its gradients are successively reduced as the optimum is approached enabling a saving of effort in gradient computations. The program, called MINOPT [4], can also be restarted efficiently from any point instead of having to repeat the entire process. Convergence of the basic algorithm has been proved [6].

THE p ALGORITHM

An Implementation of Work by Chu [8]

p_{r+1} is set to cp_r , where $c > 1$. ξ is kept constant. We let $\phi_0^r = \check{\phi}^r$, and for $r > 1$

$$\phi_j^r = \frac{c^j \phi_{j-1}^r - \phi_{j-1}^{r-1}}{c^j - 1}, \quad j = 1, \dots, n_0 \quad (8)$$

where $n_0 = \min[r-1, n]$, ϕ_j^r signifies the j th-order estimate of $\check{\phi}$ after r minima have been obtained, and n is the highest order of extrapolation.

The next minimum is estimated by solving for $n_0 \geq 1$

$$\phi_{j-1}^{r+1} = \frac{(c^j - 1)\phi_j^{r+1} + \phi_{j-1}^r}{c^j} \quad (9)$$

for $j = n_0, \dots, 1$, using $\phi_{n_0}^{r+1} = \phi_{n_0}^r$. The next starting point is ϕ_0^{r+1} .

We take $I^{r+1} = \{i \mid \mu_i > \eta\}$ where η is preassigned and

$$\mu_i = \frac{v_i}{\sum_{i \in K} v_i} \quad (10)$$

$$v_i = \begin{cases} \left(\frac{f_i(\phi_0^{r+1}) - \xi}{M(\phi_0^{r+1}, \xi)} \right)^{q_{r+1}}, & i \in K \\ 0, & i \notin K \end{cases} \quad (11)$$

and K is chosen according to (2) with I set to I^r .

p is not increased when $r = 1$ in determining μ_i . No extrapolation can be made at this stage, so the starting point for the next minimization is the solution to the first one. In the program, called FLOPT2 [5], past solutions may be retained for future runs permitting extrapolation to be implemented immediately. Theoretical justification of the algorithm has been demonstrated [7], [8].

EXAMPLES

The unconstrained minimization method throughout was a recent quasi-Newton method [9], and the computer used was a CDC 6400.

We consider the same two starting points (leading to Problems 1 and 2, see Table I) and sample points for the optimization in the minimax sense of a three-section 100-percent relative-bandwidth 10:1-transmission-line transformer as in Charalambous and Bandler [2, table I]. The effort to reach or exceed a reflection coefficient of 0.19729 (optimal to 5 figures) is compared taking f_i as the modulus of the reflection coefficient. We take as variables the lengths l_i and characteristic impedances Z_i . About $\frac{1}{3}$ to $\frac{1}{2}$ of the response evaluations used by the Charalambous-Bandler algorithms [2], tabulated in Table I, are required. Increasing p in the ξ algorithm gave poorer results. The p algorithm appears relatively insensitive to the sequence of p used.

Table II shows details of the progress of the p algorithm on Problem 1 and Table III shows the corresponding progress of the ξ algorithm.

A bandpass filter of symmetrical structure [10], the first four components of which are a unit element followed by a shunt shorted stub, a series open stub, and a shunt shorted stub, is considered next, with specifications of 0.1 dB from 1.0875 to 3.2625 GHz (passband) and 50 dB at 0.6 and 3.75 GHz. The initial normalized characteristic impedances were taken as 0.63, 0.33, 1.27, 0.26, 1.27, 0.33, and 0.63. The lengths were fixed at normalized values of 1. The functions f_i were set to $\pm$ the difference in decibels between the response and specifications, the positive sign corresponding to the

TABLE I
OPTIMIZATION OF A THREE-SECTION 10:1 TRANSFORMER OVER
100-PERCENT RELATIVE BANDWIDTH

Effort ^a required to reach or exceed a reflection coefficient of 0.19729 (optimal to 5 figures)							
Method	Problem 1				Problem 2		
	Parameter ξ or p	Function Evaluations	Sample Points	Response Evaluations	Function Evaluations	Sample Points	Response Evaluations
ξ -algorithm	0.1	28	11	308	19	11	209
(lower bound)	0.18846	16	7	112	16	7	112
$p = 2$	0.19730	45	4	180	53	4	212
$\xi^0 = 0$	0.19729	89	—	600	88	—	533
p -algorithm	8	39	11	429	29	11	319
3rd order	48	17	8	136	18	8	144
extrapolation	288	15	4	60	14	4	56
$\xi = 0$	1728	12	4	48	11	4	44
$n = 0.001$	—	83	—	673	72	—	563
Charalambous -	Alg. 1	165	11	1815	105	11	1155
Bandler [2]	Alg. 2	155	11	1705	95	11	1045

^a Does not include response evaluations to determine sample points to be used.

TABLE II
PROGRESS OF THE p ALGORITHM ON PROBLEM 1

r	Value of p_r	rth optimum ^a	Extrapolated Solution	Max. reflection coefficient	
				rth optimum	at extrapolated solution
1	8	.98828	same	.21017	.21017
		1.62868			
		1.00004			
		3.16228			
		.98828			
		6.13993			
2	48	.99833	1.00035	.19838	.19863
		1.63478	1.63600		
		.99991	.99988		
		3.16228	3.16228		
		.99833	1.00035		
		6.11703	6.11246		
3	288	.99973	1.00000	.19747	.19732
		1.63472	1.63467		
		.99999	1.00000		
		3.16228	3.16228		
		.99973	1.00000		
		6.11726	6.11744		
4	1728	.99995	1.00000	.19732	.19729
		1.63471	1.63471		
		1.00000	1.00000		
		3.16228	3.16228		
		.99995	1.00000		
		6.11730	6.11730		

^a Parameter vector $[l_1/l_q Z_1 l_2/l_q Z_2 l_3/l_q Z_3]^T$ where l_q is the quarter-wavelength at center frequency.

TABLE III
PROGRESS OF THE ξ ALGORITHM ON PROBLEM 1

r	Value of ξ^T	rth optimum ^a	Max. reflection coefficient
1	.1	.97238	.25530
		1.59720	
		.98791	
		3.16228	
		.97238	
		6.26097	
2	.18846	.99709	.19929
		1.63451	
		1.00013	
		3.16228	
		.99709	
		6.11804	
3	.19730	1.00000	.19729
		1.63471	
		1.00000	
		3.16228	
		1.00000	
		6.11730	

^a Parameter vector $[l_1/l_q Z_1 l_2/l_q Z_2 l_3/l_q Z_3]^T$ where l_q is the quarter-wavelength at center frequency.

TABLE IV
LOWER BOUNDS FOR THE SEVEN-SECTION FILTER

Stopband Specification (dB)	Passband Specification 0.1 dB		
	Value of p_2		
	Value of ξ_0		
First Maximum Error (dB)	Predicted Lower Bound (dB)	Next Maximum Error (dB)	
50	-0.0256	-0.0283	-0.0282
55	0.1430	0.1154	0.1160
60	0.6211	0.4954	0.4986
65	1.5486	1.3148	1.3195

passband, the negative sign to the stopband. Twenty-one uniformly spaced passband points were considered but, due to symmetry, only the first ten were actually used.

We let $\xi = 0$, $\eta = 0.0001$, $p_1 = 2$, $p_2 = 12$, $p_3 = 72$, and $p_4 = 432$. Third-order extrapolation was used leading to 73, 16, 14, and 12 function evaluations per optimization. The sample points were, respectively, 11, 11, 7, and 6 for a total of 1149 response evaluations. The final extrapolated solution gives characteristic impedances of 0.606 458, 0.303 062, 0.722 085, 0.235 612, 0.722 085, 0.303 062, and 0.606 458 (symmetrical as expected). The 8 passband ripples as

evaluated at the 21 points fell within 0.065 30–0.065 31 dB. The stopband responses were 50.0347 dB. Central processing unit (CPU) time on the CDC 6400 was 5 s. About half this time would be expected if characteristic impedance symmetry were exploited.

The ξ algorithm generated the results of Table IV employing the actual ripple maxima in the passband as found by quadratic approximations. The lower bound estimation process works whether the specifications are violated or satisfied, yielding an immediate indication of how good a design in the minimax sense one can expect from the results of only one least squares approximation. This table

also shows, by comparing columns 3 and 4, that only two optimizations with $p = 2$ of the ξ algorithm yield, for engineering purposes, essentially equal-ripple responses. For a 50-dB specification the final solution has characteristic impedances¹ of 0.606 595, 0.303 547, 0.722 287, 0.235 183, 0.722 287, 0.303 547, and 0.606 595 with deviations from specifications of $-0.028\ 245$ (equal to at least 5 figures).

CONCLUSIONS

Two new algorithms and related results for the least p th approach to minimax design have been presented. Documented computer programs are available from J. W. Bandler at a nominal charge. A more detailed presentation of this material is also available [11].

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¹ Actually symmetrical to at least the accuracy of the CDC 6400.

Analysis of the Transient Temperature Distribution in a Stripline with Triple-Layer Dielectric

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Abstract—The transient temperature distributions in the cross section of a stripline with triple-layer dielectric substrate are found by employing the finite element method. The calculations for three cases of different depths of center conductor considered as heat source are shown.

For each case, the calculated temperature distributions are shown at $t = 10$ s when the temperature variation has a large gradient in time and at the steady state.

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With the integralization of electric devices and the appearance of high-power semiconductor devices, miniaturization of transmission systems has become necessary. In dealing with such systems, it is of considerable importance to confirm the rise and distribution of temperatures appearing in the operation of the devices in as much as they are in a risk of thermal destruction and thermal degeneration. For the analysis of these problems, some theoretical methods and numerical procedures have been proposed. There are, however, various difficulties in the analysis of such theoretical methods.

In the present analysis, the finite element method based on the variational method was used because of its advantage in dealing with the complicated contours as well as composite media. In relation to the heat conduction equation in a two-dimensional case, the functional is defined as [1]

$$\chi = \iint \left\{ \frac{1}{2} k \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] - \left(q - C \frac{\partial T}{\partial t} \right) T \right\} dx dy \quad (1)$$

where T is the temperature, C the heat capacity, k heat conductivity, and q the rate of heat generation. The temperature rise and distribution can be obtained by finding the function T by which the functional χ is made stationary. To carry out the preceding method, the domain is divided into many triangular elements, χ is differentiated with respect to T , the derivative is set equal to zero. The resulting equation is thus given by

$$\left\{ \frac{\partial \chi}{\partial T_n} \right\} = \sum \left\{ \frac{\partial \chi}{\partial T_n} \right\}^e = [H]\{T\} + [P] \left\{ \frac{\partial T}{\partial t} \right\} - \{K\} = 0 \quad (2)$$

where $[H]$ is the heat conductivity matrix, $[P]$ is the heat capacity matrix, and $\{K\}$ is a vector which expresses the distribution of heat sources. Applying the trapezoidal approximation for the derivative with respect to time, the following difference equation was obtained for all nodal temperatures in a matrix form

$$\left([H] + \frac{2}{\Delta t} [P] \right) \{T\}_t = [P] \left(\left\{ \frac{\partial T}{\partial t} \right\}_{t-\Delta t} + \frac{2}{\Delta t} \{T\}_{t-\Delta t} \right) + \{K\}_t \quad (3)$$

To illustrate the correctness of the method, we employed a simple problem where heat source q ($=100$ W/cm³) distributes uniformly in the square column of alumina with infinite length under the Newton cooling condition. The temperature rise at the center of the column obtained by both the exact analytical solution and by the finite element method are shown in Fig. 1, and difference between the two methods is within one percent.

Then the temperature characteristics are calculated for striplines with triple-layer dielectric media. The analytical model is shown in Fig. 2 where $H = W_2 = 0.1$ cm, $a_1 = a_2 = a_3 = H/3$, $W_1 = 10a_1$, and $b = 0.001$ cm. For symmetry, the right half-plane is considered. The center medium is alumina, the heat source material is copper, and other media are glass. The respective thermal constants are shown in Table I. The rate of heat generation per unit volume q is 10^4 W/cm³. The boundary condition at the surface where $x = 0$ is

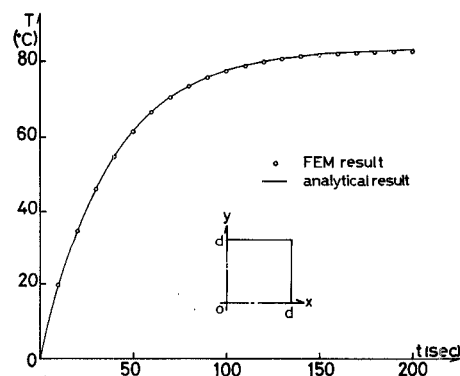


Fig. 1. Example 1: Geometry of the problem and the temperature rise at center of heat source. $d = 0.5$ mm.